Study of Grassmann Algebra with Differential Forms

Md. Anowar Hossain[1] and Md. Abdul Halim[2]

[1]Department of Natural Sciences, Stamford University Bangladesh
Dhaka-1217, Bangladesh
Email: hossain_anowar45@yahoo.com
[2]Department of Mathematics ,International University of Business Agriculture and Technology
Dhaka 1230,Bangladesh
Email: halimdu226@gmail.com

Abstract. The aim of this paper is devoted to the study of an exterior algebra (Grassmann Algebra) and briefly discusses differential forms. Using this we have developed some important theorems and propositions. Finally we also represent the integration of differential forms with the help of Grassmann algebra.

Keywords: Grassmann algebra, Exterior product.

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1. Introduction

Exterior algebra [1] and differentials forms are two important sections in differential geometry [7]. In mathematics, the exterior product or wedge product [3] of vectors is an algebraic construction used in Euclidean geometry to study areas, volumes, and their higher-dimensional analogs. The exterior product of two vectors u and v, denoted by $u \wedge v$, is called a bivector. The magnitude of $u \wedge v$ can be interpreted as the area of the parallelogram with sides u and v, which in three-dimensions can also be computed using the cross product of the two vectors. Also like the cross product, the exterior product is anticommutative, meaning that $u \wedge v = -v \wedge u$ for all vectors u and v. Tensors products are not at all necessary for the understanding or use of Grassmann algebra. As we shall it is possible to build Grassmann algebra using tensors products as a tool. We develop the elementary theory of Grassmann algebra on an axiomatic basis. Finally we discuss the integration of differential forms by using exterior algebra.

Definition 1. The exterior algebra $\wedge(V)$ over a vector space V over a field K is defined as the quotient algebra of the tensor algebra by the two-sided ideal I generated by all elements of the form $x \otimes x$ such that $x \in V$. Symbolically,

$$\wedge(V) \coloneqq T(V)/I$$

the wedge product $'\wedge'$ of two elements of $\wedge(V)$ is defined by $\alpha \wedge \beta = \alpha \otimes \beta \ (mod\ I)$

The exterior algebra was first introduced by Hermann Grassmann in 1844.

Axioms of Grassmann Algebra:

1. The Grassmann product is associative that is,

$$(f \wedge g) \wedge h = f \wedge (g \wedge h)$$

2. The Grassmann product is multilinear that is,

$$v_1 \wedge \ldots \wedge (\alpha^1 u_1 + \alpha^2 u_2 \wedge \ldots \wedge v_r) = \alpha^1(v_1 \wedge \ldots \wedge u_1 \wedge \ldots \wedge) + \alpha^2(v_2 \wedge \ldots \wedge u_2 \wedge \ldots \wedge)$$

3. The product is nilpotent that is, any $v \in V$, $v \wedge v = 0$

4. The set of all products $e_{i_1} \wedge \ldots \wedge e_{i_r}$ is linearly independent.

2. The exterior power

The k-th exterior power of V, denoted $\wedge^k(V)$, is the vector subspace of $\wedge(V)$ spanned by elements of the form

$$x_1 \wedge x_2 \wedge \ldots \wedge x_k \ , \quad x_i \in V, i = 1,2, \ldots, k$$

If $\alpha \in \wedge^k(V)$, then α is said to be a k-multivector. If, furthermore, α can be expressed as a wedge product of k elements of V, then α is said to be decomposable.

For example, in$\mathbb{R}^4$, the following 2-multivector is not decomposable: $\alpha = e_1 \wedge e_2 + e_3 \wedge e_4$

This is in fact a symplectic form, since $\alpha \wedge \alpha \neq 0$.

We can express the 2-form $dx \wedge dy$ in polar coordinates by setting $x = rcos\theta,\ y = rsin\theta$ we obtain

$$dx \wedge dy = rdr \wedge d\theta$$

3. Exterior derivative of a *k*-form

The exterior derivative [8] is defined to be the unique R-linear mapping from *k*-forms to (*k*+1)-forms satisfying the following properties:

1. df is the differential of f for smooth function f.
2. $d(df) = 0$ for any smooth function f.
3. $d\,(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p(\alpha \wedge d\beta)$ where α is a p-form. That is to say, d is a derivation of degree 1 on the exterior algebra of differential forms.

The second defining property holds in more generality: in fact, $d(d\alpha) = 0$ for any k $-$form α.The third defining property implies as a special case that if f is a function and α a k-form, then $d(f\alpha) = df \wedge \alpha + f \wedge d\alpha$ because functions are forms of degree 0.

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Theorem 2 (Poincare's Lemma). [9] $d^2 = 0$, that is for any exterior differential form $\omega, d(d\omega) = 0$.

Theorem 3. Suppose ω is a differential 1-form on a smooth manifold M. X and Y are smooth tangent vector fields on M. Then

$$< X \wedge Y, d\omega > = X < Y, \omega > - Y < X, \omega > - < [X, Y], \omega >$$

Proof. Given $< X \wedge Y, d\omega > = X < Y, \omega > - Y < X, \omega > - < [X, Y], \omega >$ (1)
since both sides of (1) are linear with respect to ω, we may assume that ω is a monomial
$\omega = g\, df$; where f and g are smooth functions on M

$$\Rightarrow d\omega = dg \wedge df$$

L.H.S: $< X \wedge Y, d\omega >$

$$= < X \wedge Y, \quad dg \wedge df >$$

$$= \begin{vmatrix} < X, dg > & < X, d > \\ < Y, dg > & < Y, d > \end{vmatrix}$$

$$= \begin{vmatrix} Xg & Xf \\ Yg & Yf \end{vmatrix} = Xg.Yf - Xf.Yg$$

R.H.S: $X < Y, \omega > - Y < X, \omega > - < [X, Y], \omega >$

$$= X < Y, g\, df > - Y < X, g\, df > - < [X, Y], g\, df >$$

$$= X(g\, Yf) - Y(g\, Xf) - g[X, Y]f$$

$$= Xg.Yf + g\, XYf - Yg.Xf - g\, YXf - g\, XYf + g\, YXf = Xg.Yf - Xf.Yg$$

Therefore L.H.S = R.H.S
This complete the proof of the theorem □

Example 1. For a 1-form $\sigma = u\, dx + v\, dy$ defined over $\mathbb{R}^2$. We have, by applying the above formula to each term (consider $x^1 = x$ and $x^2 = y$) the following sum,

$$d\sigma = \left(\sum_{i=1}^{2} \frac{\partial u}{\partial x^i} dx^i \wedge dx\right) + \left(\sum_{i=1}^{2} \frac{\partial v}{\partial x^i} dx^i \wedge dy\right)$$

$$= \left(\frac{\partial u}{\partial x} dx \wedge dx + \frac{\partial u}{\partial y} dy \wedge dx\right) + \left(\frac{\partial v}{\partial x} dx \wedge dy + \frac{\partial v}{\partial y} dy \wedge dy\right)$$

$$= 0 - \frac{\partial u}{\partial y} dy \wedge dx + \frac{\partial v}{\partial x} dx \wedge dy + 0$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx \wedge dy.$$

Example 2. Suppose the Cartesian coordinates in $\mathbb{R}^3$ are given by (x, y, z).

1) If f is a smooth function on $\mathbb{R}^3$, then $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$.

The vector formed by its coefficients $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is the gradient of f, denoted by $grad\, f$.

2) Suppose $a = Adx + Bdy + Cdz$, where A, B, C are smooth functions on $\mathbb{R}^3$. Then $da = dA \wedge dx + dB \wedge dy + dC \wedge dz$

$$= \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy.$$

Let X be the vector (A, B, C), then the vector

$$\left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}, \frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}, \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)$$

formed by the coefficients of da is just the $curl$ of the vector field X, denoted by $curl\, X$.

3) Suppose $a = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$. Then

$$da = \left(\frac{\partial A}{\partial x}, \frac{\partial B}{\partial y}, \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz$$

$$= div\, X\, dx \wedge dy \wedge dz$$

where $div\, X$ means the divergence of the vector field

$$X = (A, B, C).$$

From theorems, two fundamental formulas in a vector calculus follow immediately. Suppose f is a smooth function on $\mathbb{R}^3$ and X is a smooth tangent vector field on $\mathbb{R}^3$. Then

$$\begin{cases} curl\,(grad\, f) = 0 \\ div\,(curl\, X) = 0 \end{cases}$$

4. Integration of Differential Forms

The calculus of differential forms [2], [6] provides a convenient setting for integration on manifolds, as we explain in this section due to the efficient way it keeps track of change of variables.
A k −form β on an open set $G \subset \mathbb{R}^n$has the for

$$\beta = \sum_j b_j(x) dx_{j_1} \wedge \ldots \wedge dx_{j_k} \qquad (1)$$

Here $j = (j_1, \ldots, j_k)$ is a k −multi-index. We write $\beta \in \wedge^k(G)$. The wedge product used in (1) has the anti-commutative property

$$dx_l \wedge dx_m = -dx_m \wedge dx_l$$

So that if $\sigma$is a permutation of $\{1, \ldots, k\}$ we have

$$dx_{j_1} \ldots \wedge dx_{j_k} = (sgn\, \sigma) dx_{j_{\sigma(1)}} \wedge \ldots \wedge dx_{j_{\sigma(k)}} \qquad (2)$$

In particular, an n −form α on $\Omega \subset \mathbb{R}^n$ can be written

$$\alpha = A(x) dx_1 \wedge \ldots \wedge dx_n$$

If $A \in L^1(G, dx)$ then we write

$$\int_G \alpha = \int_G A(x) dx \qquad (3)$$

the right side being the usual Lebesgue integral.
Suppose now $\Omega \subset \mathbb{R}^n$ is open and there is a C^1 diffeomorphism $F: \Omega \to$ G. We define the pull back $F^*\beta$ of the k −form in (1) as

$$F^*\beta = \sum_j b_j\left(F(x)\right)\left(F^* dx_{j_1}\right) \wedge \ldots \wedge \left(F^* dx_{j_k}\right) \qquad (4)$$

where

$$F^* dx_j = \sum_l \frac{\partial F_j}{\partial x_l} dx_l$$

If $B = (b_{lm})$ is an $n \times n$ matrix then by (2) and the formula for the determinant gives

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$$\left(\sum_m b_{1m}\,dx_m\right)\wedge\left(\sum_m b_{2m}\,dx_m\right)\wedge\ldots\wedge\left(\sum_m b_{nm}\,dx_{nm}\right)$$
$$=\left(\sum_\sigma (sgn\,\sigma)\,b_{1\sigma(1)}b_{2\sigma(2)}b_{n\sigma(n)}\right)dx_1\wedge\ldots\wedge dx_n$$
$$=(\det B)\,dx_1\wedge\ldots\wedge dx_n$$

Hence if $F:\Omega\to\mathrm{G}$ is a C^1 map and α is an $n-$form on Gas in (4) then

$$F^*\alpha=\det D\,F(x)\;A\big(F(x)\big)dx_1\wedge\ldots\wedge dx_n$$

This formula is especially significant in light of the change of variable formula

$$\int_G A(x)dx=\int_\Omega A\big(F(x)\big)\;|\det D\,F(x)|dx \qquad (5)$$

The only difference between the right side of (5) and $\int_\Omega F^*\alpha$ is the absolute value sign around $\det D\,F(x)$. We say a C^1 map $F:\Omega\to\mathrm{G}$ is orientation preserving when $\det D\,F(x)>0$ for all $x\in\Omega$.

Proposition 1. If $F:\Omega\to\mathrm{G}$ is a C^1 orientation preserving diffeomorphism and α an integrable $n-$ form on G then

$$\int_G\alpha=\int_\Omega F^*\alpha$$

Proof. The wedge product of dx_l's extends to a wedge product on form as follows. If $\beta\in\wedge^k(G)$ has the form (1) and if

$$\alpha=\sum_i\alpha_i\,(x)dx_{i_1}\wedge\ldots\wedge dx_{i_l}\in\wedge^l(G)$$

define

$$\alpha\wedge\beta=\sum_{i,j}\alpha_i\,(x)b_j(x)\,dx_{i_1}\wedge\ldots\wedge dx_{i_l}\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_l}$$

in $\wedge^{k+l}(G)$, it follows that $\alpha\wedge\beta=(-1)^{kl}\beta\wedge\alpha$

It is also readily verified that $F^*(\alpha\wedge\beta)=(F^*\alpha)\wedge(F^*\beta)$. Another important operator on forms is the exterior derivative $d:\wedge^k(G)\to\wedge^{k+1}(G)$ defined as follows. If $\beta\in\wedge^k(G)$ is given by (3) then

$$d\beta=\sum_{j,l}\frac{\partial b_j}{\partial x_l}\,dx_l\wedge dx_{j_1}\wedge\ldots\wedge dx_{j_k}$$

If $\beta\in\wedge^k(G)$ and $F:\Omega\to\mathrm{G}$ is a smooth map. Now

$$dF^*\beta$$
$$=\sum_{j,l}\frac{\partial}{\partial x_l}\Big(b_j\circ F(x)\Big)\,dx_l\wedge\big(F^*dx_{j_1}\big)\wedge\ldots\wedge\big(F^*dx_{j_k}\big)$$
$$+\sum_{j,v}(\pm)\,b_j(F(x))(F^*dx_{j_1})\wedge\ldots\wedge\big(F^*dx_{j_v}\big)\wedge\ldots\wedge\big(F^*dx_{j_k}\big)$$

Now pull back gives directly that

$$F^*dx_i=\sum_l\frac{\partial F_i}{\partial x_l}\,dx_l=dF_i$$

and hence $d(F^*dx_i)=ddF_i=0$, so only first sum in (A) contributes to $dF^*\beta$. Meanwhile

$$F^*d\beta$$
$$=\sum_{j,m}\frac{\partial b_j}{\partial x_m}(F(x))(F^*dx_m)\big(F^*dx_{j_1}\big)\wedge\ldots\wedge\big(F^*dx_{j_k}\big)$$

so we have

$$\sum_l\frac{\partial}{\partial x_l}\Big(b_j\circ F(x)\Big)\,dx_l=\sum_m\frac{\partial b_j}{\partial x_m}(F(x))F^*dx_m$$

which in turn follows from the chain rule.

If $H:M\to\mathbb{R}^m$ is a smooth map and γ is a $k-$form on$\mathbb{R}^m$ then there is well defined $k-$form $\gamma=H^*\gamma$ on M.represented in such coordinate charts $\beta_J=(H\circ F_j)^*\gamma$. Similarly if $\beta-$is a $k-$form on Mas defined above and $H:U\to M$is smooth with $U\subset\mathbb{R}^m$ open then $H^*\beta$ is a well defined $k-$form on U.

We define the integral of an $n-$form over an oriented $n-$dimensional manifold as follows. First $\alpha-$in an $n-$form supported on an open set $G\subset\mathbb{R}^n$ given by (4) then we define $\int_G\alpha$b.

More generally, if $M$is an $n-$dimensional manifold with an orientation say the image on an open set $G\subset\mathbb{R}^n$ by $\varphi:G\to M$ carrying the natural orientation of G we can set

$$\int_M\alpha=\int_G\varphi^*\alpha$$

For an $n-$form α on M. If it takes several coordinates patches to cover M define $\int_M\alpha$ by writing α as a sum of forms, each supported on one patch.

We need to show that this definition of $\int_M\alpha$ is a independent of the choice of coordinate system on M. Thus suppose $\varphi:G\to U\subset M$ and $\psi:\Omega\to U\subset M$ are both coordinate patches so that $F=\psi^{-1}\circ\varphi:G\to\Omega$ is an orientation preserving diffeomorphism. We need to check that if α is an $n-$form on M supported on U, then

$$\int_G\varphi^*\alpha=\int_\Omega\psi^*\alpha=\int_G F^*(\psi^*\alpha)$$

Thus the integral of an $n-$form over an oriented $n-$dimensional manifold is well defined.

Proposition 2. Given a compactly supported $(k-1)-$form β of class C^1 on an oriented $k-$dimensional surface $\bar{M}$ (of class C^2) with boundary ∂M with its natural orientation

$$\int_M d\beta=\int_{\partial M}\beta \qquad (6)$$

Proof. Using a partition of unity and invariance of the integral and the exterior derivative under coordinate transformations it suffices to prove this $\overline{M}=\{x\in\mathbb{R}^k:x_1\le 0\}$. In that case we will be able to deduce (6) from the fundamental theorem of calculus. If $\beta=b_j(x)dx_1\wedge\ldots\wedge dx_j\wedge\ldots\wedge dx_k$ with $b_j(x)$ of bounded support, we have

$$d\beta=(-1)^{j-1}\frac{\partial B_j}{\partial x_j}dx_1\wedge\ldots\wedge dx_n$$

If $j>1$ we have

$$\int_M d\beta=(-1)^{j-1}\int\left\{\int_{-\infty}^{\infty}\frac{\partial B_j}{\partial x_j}dx_j\right\}dx^l=0$$

And also $k^*\beta=0$, where $k:\partial M\to\bar{M}$ is the inclusion. On the other hand for $j=1$ we have

$$\int_M d\beta=\int\left\{\int_{-\infty}^{\infty}\frac{\partial B_1}{\partial x_1}dx_1\right\}dx_2\wedge\ldots\wedge dx_k$$
$$=\int b_1(0,x^l)\,dx^l=\int_{\partial M}\beta$$

This proves Stokes' formula.

Proposition 3. [4] There is no smooth retraction $\varphi: B \to S^{n-1}$ of the close unit ball B in $\mathbb{R}^n$ onto its boundary S^{n-1}.

Proposition 4. [4] If $F: B \to$ B is a continuous map on the closed unit ball in $\mathbb{R}^n$, then F has a fixed point.

Theorem 4. For any $\omega \in \Omega^k(M)$ the formula

$$d\omega(X_1, \dots, X_{K+1}) = \sum_i (-1)^{i-1} X_i \left(\omega(X_1, \dots, \hat{X}, \dots, X_{k+1})\right) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], (X_1, \dots, \hat{X_i}, \dots, \hat{X_j}, \dots, X_{k+1})$$

defines a $(k+1)$-form $d\omega \in \Omega^{k+1}(M)$.

Proof. To show that $d\omega$ is a $(k+1)$ −form we need to show that
$d\omega$ is anti-symmetric i.e. for any $r < s$

$$d\omega(X_1, \dots, X_r, \dots, X_s, \dots, X_{k+1}) = -d\omega(X_1, \dots, X_r, \dots, X_s, \dots, X_{k+1})$$

$d\omega$ is multi-linear at each point ,i.e. $d\omega$ is $C^\infty(M)$ linear on M. Note that $d\omega$ is obviously $\mathbb{R}$ −linear. So for any $f \in C^\infty(M)$

$$d\omega(fX_1, X_2, , \dots, X_{k+1}) = fd\omega(X_1, \dots, X_{k+1})$$

This can be checked by a direct computation:

$$\begin{aligned} d\omega(fX_1, X_2, , \dots, X_{k+1}) &= fX_1\left(\omega(X_2, \dots, X_{k+1})\right) \\ &+ \sum_{i>1} (-1)^{i-1} X_i \left(\omega(fX_1, \dots, \hat{X_i}, \dots, X_{k+1})\right) \\ &+ \sum_{i>1} (-1)^{i+j} \omega([fX_1, X_i], (X_2, \dots, \hat{X_i}, \dots, X_{k+1}) \\ &= fd\omega(X_1, \dots, X_{k+1}) \\ &+ \sum_{i>1} (-1)^{i-1} (X_i f) \left(\omega(X_1, \dots, \hat{X_i}, \dots, X_{k+1})\right) \\ &- \sum_{i>1} (-1)^{i-1} (X_i f) \left(\omega(fX_1, \dots, \hat{X_i}, \dots, X_{k+1})\right) \\ &= fd\omega(X_1, \dots, X_{k+1}) \end{aligned}$$

5. Geometrical description of Differential forms:

The geometrical notion of the gradient of the function, such as the temperature in a room, the ordinary 3-dimensional vector $\nabla T(\vec{r})$ defines a vector field throughout the room, which is usually described by saying that it is the direction of greatest change of T, at the particular point with coordinates $\vec{r}$. However the surfaces of constant value of the temperature T to which the the direction ∇T is perpendicular. These are surfaces much like equipotentials for electrostatics, where the temperature does not change. It is altogether plausible that the surface on which the function does not change is the one that is perpendicular to the direction of its greatest change.
We therefore take the analytic idea that df is the generalization of the gradient of f, i.e., ∇f. As in the case of tangent vectors and directional derivatives, the comparison is reasonable since the two quantities have the same components, only the basis vectors look different. Again the new basis vectors for df are defined locally at each single point P on the manifold [5]. Then we may say that geometrically it corresponds to a local view of the surfaces of constant values for f. Since they are locally defined over an m −dimensional manifold, surfaces of constant values for some function, say the temperature T, are $(m-1)$ −ddimensional surfaces so called hyper surfaces since they are only one less dimension than the entire space. The simplest 1 −form dT is an algebraic representation of the set of hyper surfaces of the constant value for T at the point.

REFERENCES

[1] H. Flanders', *Differential Forms with Applications to the Physical Sciences*, Academic Press, 1962.
[2] J.C. Islam, *Modern Differential Geometry for Physicist*, World Scientific Publishing Co. Pt. Ltd., 1989.
[3] M.A. Hossain and M.S. Ali, Study on Exterior Algebra Bundle and Differential Forms, *Annals of Pure and Applied Mathematics*, Vol. 5, No.2, 2014, 198-207.
[4] M. Nakahara, *Geometry, Topology and Physics*, IOP Publishing Limited, 1990.
[5] M.S. Ali, K.M. Ahmed, M.R. Khan and M.M. Islam, Exterior algebra with differential forms on manifolds, *Dhaka Univ. J.Sci.*, 60(2) (2012) 247-252.
[6] M. Shigeyuki, *Geometry of Differential Forms*, Japanese by Iwanami Shoten Publishers, Tokyo, 1998.
[7] P.P. Gupta and G.S. Malik, *Tensors and Differential Geometry*, Pragati Prakashan, Meerut, U.P., India, 2000.
[8] S.P. Novikov and A.T. Fomenko, *Basic Elements of Differential Geometry and Topology*, 1995.
[9] S.S. Chern, W.H. Chen and K.S. Lam, *Lectures on Differential Geometry*, World Scientific Publishing Co. Pt. Ltd., 2000